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Birational Extensions of Regular Local Rings and Local UFDs

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INTRODUCTION

Abhyankar's generalization [1, Theorem 3] of Zariski's Local Factorization Theorem [13, Lemma, p. 538] states that for R and S 2-dimensional regular local rings (RLRs) such that $R \subseteq S \subseteq$ quotient field (R) (which we henceforth abbreviate as q.f.(R)), there exists a unique finite sequence of 2-dimensional RLRs, $R = R_0 \subseteq R_1 \subseteq \cdots \subseteq R_{n+1} = S$, between R and S such that, moreover, each R_i is a quadratic transform of R_{i-1} (i.e., is obtained from R_{i-1} by blowing up its maximal ideal). Thus one has a good understanding of the relationship between R and S and many invariants, if known for R, can be easily computed for S.

In higher dimensions, in general, it is no longer possible to factor a pair of birationally dominating RLRs $R \subseteq S$ as above via blow-ups of the maximal ideal nor even via blow-ups of regular ideals (called "monoidal transformations") (see [9, Corollary 4.5; 11, Example 3.2]). Nonetheless, one is led to ask what sorts of factorizations via RLRs of the same dimension may be possible for R and S as above, where $\dim R = \dim S =: d \geqslant 3$. One can also ask whether there exists some analogue of the Local Factorization Theorem of Zariski and Abhyankar for certain singular local rings, in particular for unique factorization domains (UFDs). More precisely, we consider the following questions:

QUESTION 1 (Uniform bound). *If $R \subset S \subset$ q.f.(R) ("$\subset$" denotes strict containment) are d-dimensional local UFDs ($d \geqslant 2$), does there exist a number $B(R, S)$ depending only on R and S such that, for every chain $R \subset R_1 \subset \cdots \subset R_n \subset S$ of local UFDs, $n \leqslant B(R, S)$?*

QUESTION 2 (Minimal extensions). *If $R \subset S \subset$ q.f.(R) are d-dimensional RLRs ($d \geqslant 3$), and if $R \subset T \subset S$ with T a RLR minimal over R (i.e., there are*

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no RLRs properly between T and R), then what structure can T possess as an extension of R? In particular, can one classify such minimal extensions?

QUESTION 3 (Maximally Regular). *Given an extension $R \subset T \subset S$ as above, if T is maximal as a RLR under S (i.e., there are no RLRs properly between T and S), then what structure can S possess as an extension of T?*

A few comments concerning these questions are in order. Question 1 has an affirmative answer if R and S are RLRs and R is excellent [6, Corollary 1.11]. It has a negative answer for many natural generalizations of regularity (e.g., Gorenstein, rational singularity), even in dimension 2 [4, Examples 4.9 and 5.2]. However, the case of birational local UFDs is still intriguing because Lipman has shown [7, Proposition (3.1)] that any 2-dimensional local UFD which birationally dominates a 2-dimensional RLR must itself be regular, so Question 1 has an affirmative answer for such local UFDs. Section 1 below provides some more evidence that local UFDs are an interesting class of singularities for this question. Question 2 is much vaguer and perforce much more difficult to answer. It was shown by Sally and Shannon that simple monoidal transforms are always minimal [9, Theorem 5.1; 11, Example 5.4] and by Sally [9] that another type of extension obtained by blowing up a certain height 2 non-regular complete intersection prime of R (dubbed "Sally extensions" by the author [4, p. 523]) are minimal. Very little is known otherwise, although it is known that such a T must always exist [5, Corollary]. Of course, Questions 2 and 3 are essentially the same, differing only in viewpoint. Question 3 is a variant on some recent work done by Huneke and Sally [3] in which they use the concept of maximal regularity inside a 2-dimensional normal local domain S in order to elucidate the ideal structure of S.

The present paper presents some partial answers to the questions above. In Section 1 we show an affirmative answer to Question 1 for certain extensions S of R (both local UFDs). The uniform bound we find is of particular interest because, unlike most of the previous arguments known in this area, it holds even in the case that $\dim R > \dim S$. It also differs from previous proofs in that it does not rely on Zariski's Main Theorem (ZMT) as it occurs in Nagata's book [8, (37.4)], but instead relies on Zariski's concept of the irregular locus of a birational correspondence. In Section 2 we tackle Questions 2 and 3 above in the simplest possible case, namely, when R and S are 3-dimensional and S is a quadratic transform of R. We are able to show, in this case, that there can exist at most one RLR strictly between R and S (this has already been shown in [6, Prop. 2.3] under the further assumption that R is excellent) and that for any such RLR T, S must be a simple monoidal transform of T (provided that R and S have the same residue field). The question of whether T must in turn be a simple monoidal transform of R has unfortunately eluded proof. Finally, in Section 3 we make some further attempts to affirmatively answer Question 1 for arbitrary RLRs of the same dimension (i.e., without the assumption that R is excellent). Our most striking result here is a nice uniform bound on chains between a RLR of arbitrary dimension and a monoidal transform of it, without any excellence assumption. These results are mainly of interest in that they use very different techniques from those in [6].

We remark on notation. All rings are commutative with non-zero identity, and local rings are Noetherian. There are three specific classes of extensions of RLRs about which one has some understanding at present, namely, quadratic transforms, monoidal transforms, and Sally extensions. The most classical of the three are quadratic transforms: we say that the pair $(S, \mathcal{N}) > (R, \mathcal{M})$ of d-dimensional RLRs is a *quadratic transform* iff $S = R[\mathcal{M}/x]_P$, where x is a regular parameter in R and $P \in \operatorname{Spec}(R[\mathcal{M}/x])$. In dimension 2, by the Local Factorization Theorem of Zariski and Abhyankar, these are all that matter. In dimension 3 and higher, it is natural to introduce monoidal transforms: with S and E as above, S *is a monoidal transform of* R means that $S = [Q/x]_P$, where Q is generated by a subset of a minimal generating set for $\mathcal{M}$ (called a *regular system of parameters* for R, sometimes abbreviated r.s.o.p.), $x \in Q$, and P is a (necessarily maximal) prime in the finitely generated R-algebra. We say that the monoidal transform is *simple* iff Q is 2-generated. Finally, a *Sally extension* occurs when $S = R[a/b]_{\mathscr{P}}$, where $\mathscr{P}$ is a maximal ideal of the algebra, and $b \in \mathcal{M}^2$, $a \notin \mathcal{M}^2$, and a and b are nonzero relatively prime elements of R. A *spot* over R is a localization at a prime ideal of a finitely generated R-algebra. A *regular prime* of R is a prime ideal P of R such that R/P is a RLR. If S and R are quasilocal domains, the notation $S > R$ indicates that these rings have the same quotient field, and that S *dominates* R, i.e., that S contains R and the maximal ideal of S contains that of R. If I is an ideal of the ring R, we use $\operatorname{gr}(R, I)$ to denote the *associated grated ring of R with respect to I*, i.e., the ring $\bigoplus I^i/I^{i+1}$, $i \geq 0$. By the *$\mathcal{M}$-adic order valuation* of a RLR $(R, \mathcal{M})$ we mean the valuation determined on R by the rule $\operatorname{ord}_R(x) := n$, where $x \in \mathcal{M}^n \setminus \mathcal{M}^{n+1}$, and then extended to the quotient field. We say that $S > R$ is a *minimal extension* of RLRs (or local UFDs, etc.) iff S and R are RLRs (or local UFDs, etc.) and there are no RLRs (or local UFDs, etc.) properly between them.

1. Birational Unique Factorization Domains and the Irregular Locus of an Extension

In this section we see as our principal result that, for certain birational extensions of local unique factorization domain (UFDs), $S > R$, there exists a uniform bound on the length of chains of local UFDs between S and R. In order to understand such an extension, we develop the tool of the irregular locus of an extension, a concept due to Zariski [12, Definition 4]. We begin with the definition.

DEFINITION 1.1. If S and R are integral domains with $S \supseteq R$ birationally, then we shall say that $P \in \operatorname{Spec} S$ is *irregular for the extension* $S \supseteq R$ iff $S_P \neq R_{P \cap R}$. The set of all prime ideals in S which are irregular for the extension will be denoted $\operatorname{Irr}(S, R)$.

We first note that such primes always exist in the case of present interest, and we consider some of the properties possessed by the locus of primes irregular for an extension.

LEMMA 1.2. *If R is a* UFD *and if S is a Noetherian integral domain which properly contains R birationally and which is not a localization of R, then there exists a height* 1 *prime ideal of S which is irregular for the extension.*

Proof. The assumptions imply the existence of $a/b \in S \backslash R$, where $a, b \in R$, $b \neq 0$, and a and b are relatively prime in R, such that b is not a unit in S. If P is a height 1 prime of S such that $b \in P$ then $a, b \in P \cap R$, so $\operatorname{ht} P \cap R > 1$ and $P \in \operatorname{Irr}(S, R)$. ∎

LEMMA 1.3. *If* $\text{q.f.}(R) \supseteq S \supseteq T \supseteq R$ *are integral domains, where R satisfies Serre's condition R_1, and if* $P \in \operatorname{Irr}(T, R)$, *with* $\operatorname{ht} PS = 1$, *then there exists* $Q \in \operatorname{Irr}(S, R)$, $\operatorname{ht} Q = 1$, *such that $P \subseteq Q \cap T$. In particular, there exists such a Q if S is Noetherian, and both T and R are* UFDs, *with* $\operatorname{ht} P = 1$, *and* $PS \neq S$.

Proof. We have that $\operatorname{ht}(P \cap R) \geqslant 2$, for otherwise $T_p = R_{P \cap R}$, since $T_P \supseteq R_{p \cap R}$ and the latter is a DVR. Let Q be a minimal prime over PS such that $\operatorname{ht} Q = 1$. Then $Q \cap R \supseteq PS \cap R \supseteq P \cap R$, so $\operatorname{ht}(Q \cap R) \geqslant 2$, making $Q \in \operatorname{Irr}(S, R)$. ∎

LEMMA 1.4. *If $S \supseteq R$ birationally are arbitrary domains then* $\operatorname{Irr}(S, R)$ *is closed under specialization; i.e., if* $P \in \operatorname{Irr}(S, R)$, *and* $P \subseteq Q$, *where* $Q \in \operatorname{Spec} S$, *then* $Q \in \operatorname{Irr}(S, R)$.

Proof. If not, then $S_Q = R_{Q \cap R}$, which implies that $S_P = (S_Q)_{PS_Q} = (R_{Q \cap R})_{(P \cap R) R_{Q \cap R}} = R_{P \cap R}$, which is a contradiction. ∎

PROPOSITION 1.5. *If R is a UFD and if S is a finitely generated R-algebra which is a Noetherian integral domain birational to R and not a* *localization of* R, *then* $\operatorname{Irr}(S, R)$ *is a pure codimension* 1 *closed subscheme of* Spec S.

Proof. Write $S = R[a_1/b_1, ..., a_n/b_n]$, where $a_i, b_i \in R$, $b_i \neq 0$, and $\operatorname{ht}(a_i, b_i) R \neq 1$ for all i. Let $b := \prod_{i=1}^{n} b_i$. Now if P is a prime ideal of S and $b \notin P$ then $b_i \notin P$ for all i, so $b_i \in R \setminus (P \cap R)$ for all i. This implies that $S \subseteq R_{P \cap R}$, so $R_{P \cap R} = S_P$, and $P \notin \operatorname{Irr}(S, R)$. Therefore $\operatorname{Irr}(S, R) \subseteq V(b)$ (the closed subscheme defined by b in Spec S).

If b is not in any prime ideal of S then b is a unit in S, which contradicts the assumption that S is not a localization of R. Since S is Noetherian, there exists a height 1 prime of S containing b, so $V(b)$ is a pure codimension one closed subscheme of Spec S.

Now if P is a height 1 prime in S with $b \in P$, then $b_i \in P$ for some i. Thus $a_i \in P$, which implies that $\operatorname{ht}(P \cap R) \geqslant \operatorname{ht}(a_i, b_i) R = 2$, so $P \in \operatorname{Irr}(S, R)$. Moreover, if $Q \in \operatorname{Spec} S$, $b \in Q$, and $\operatorname{ht} Q > 1$, then there exists a height 1 prime $P \subseteq Q$ with $b \in P$ and therefore with $P \in \operatorname{Irr}(S, R)$. By Lemma 1.4 above, it follows that $Q \in \operatorname{Irr}(S, R)$. Hence $V(b) \subseteq \operatorname{Irr}(S, R)$, and altogether we get equality. ∎

The proof actually shows that there exists $b \in R$ such that $\operatorname{Irr}(S, R) = V(b)$. Therefore, if S as above is also a UFD, then the generic points of the irreducible components of $\operatorname{Irr}(S, R)$ are precisely those height one primes generated by the irreducible factors of b in S. We shall see below that the same is true if S is a spot over R.

COROLLARY 1.6. *If* $S \supset R$ *birationally, where* R *is a Noetherian UFD and* S *is a spot over* R *which is not a localization of* R, *then* $\operatorname{Irr}(S, R)$ *is a pure codimension* 1 *closed subscheme of* S.

Proof. Let $S := S'_{(\,)}$, where $S' := R[a_1/b_1, ..., a_n/b_n]$, with $a_i, b_i \in R$, $b_i \neq 0$, and $\operatorname{ht}(a_i, b_i) R \neq 1$ for all i, and let b be as above. Then $P \in \operatorname{Irr}(S, R)$ implies that $S'_{P \cap S'} = S_P \neq R_{(P \cap S) \cap R}$, which implies that $P \cap S' \in \operatorname{Irr}(S', R) = V(b)$ (by the proposition), so $b \in P$. Therefore $\operatorname{Irr}(S, R) \subseteq V(b)$. On the other hand, if $b \in P$, P a prime ideal, then $S_p = S'_{P \cap S} \neq R_{P \cap R}$, so $\operatorname{Irr}(S, R) = V(b)$. $\operatorname{Irr}(S, R) = V(b)$ is non-empty because S is not a localization of R. ∎

Now we are in a position to obtain our main results concerning uniform bounds on the length of chains of local UFDs between a certain fixed pair of local UFDs, $S > R$. Note that $S > R$ implies that S is not a localization of R (unless, of course, $S = R$) and this allows us to apply the lemmas above. We separate the easier case out as a lemma.

LEMMA 1.7. *If* $S > R$ *are local UFDs such that* $S = R[a_1/b, ..., a_m/b]_{(\,)}$, *where* $\operatorname{ht}(a_i, b) = 2$ $(a_i, b \in R, b \neq 0)$, bS *is prime, and* $\operatorname{ht}(bS \cap R) = 2$, *then* *there are no local UFDs $T \neq S$, $R > T > R$. Moreover, if* $\dim S = \dim R$ *then there are no local UFDs strictly between S and R.*

Proof. Suppose that T is a local UFD and $S > T > R$. If $T \neq R$, use Lemma 1.2 in order to obtain a height 1 prime ideal, P, of T such that $\operatorname{ht}(P \cap R) > 1$. Then $P \subseteq bS \cap T$ (see Lemma 1.3, and the remark before Corollary 1.6) implies that $P \cap R \subseteq bS \cap R$, which forces $P \cap R = bS \cap R$, since the latter is a height 2 prime ideal. Therefore $b \in P$ and so $P = bT$, as b irreducible in S implies that b is also irreducible in T. Moreover, we have $a_1, ..., a_m \in bT$, so $a_1/b, ..., (a_m/b) \in T$, and $T = S$.

The last statement merely amounts to the fact that for local domains of the same dimension, birational containment, and birational domination are the same thing [4; Note 2.1]. ▮

The necessity of the hypothesis "$\operatorname{ht}(bS \cap R) = 2$" in the above lemma can already be seen in the case of a quadratic transform of 3-dimensional RLRs. For example, let R be $k[X, Y, Z]_{(X, Y, Z)}$, where k is an arbitrary field, and X, Y, and Z are indeterminates. Then if $b = X$, $a_1 = Y$, $a_2 = Z$, the S occurring in the above lemma is a quadratic transform of R, and it is clear that the simple monoidal transform T of R defined by $T = R[Y/X]_{\mathcal{N} \cap R(Y/X)}$ sits properly between R and S.

THEOREM 1.8. *Let $(R, \mathcal{M})$ be a local UFD, and let $S := R[a_1/b_1 \cdots b_n), ..., a_m/(b_1 \cdots b_n)]_{\mathcal{N}}$ be another local UFD birationally dominating R, where $b_1, ..., b_n \in R$ are irreducible elements of S (not necessarily distinct). Assume that* $\operatorname{ht}(a_j, b_1 \cdots b_n) R = 2$ *for all j. If* $\operatorname{ht}(b_i S \cap R) = 2$ *for all i, then there are at most $n-1$ distinct local UFDs in any fixed chain ordered by domination strictly between S and R. Moreover, if* $\dim S = \dim R$ *then there are at most $n-1$ distinct local UFDs in any fixed chain ordered by containment strictly between S and R.*

Proof. By induction on n.

$n = 1$. This is Lemma 1.7.

$n > 1$. Let T be an arbirary local UFD, with $S > R > R$. If $T \neq R$, then we have seen above that there exists a height 1 prime P of T which is irregular for the extension $T \supseteq R$. Furthermore, $P \subseteq b_i S \cap T$ for some i, since each irreducible component of $\operatorname{Irr}(S, R)$ is exactly $V(b_i)$ for some i. Fix this i. Then $P \cap R = b_i S \cap R$, since the latter has height 2 by assumption. But b_i, $\prod_{k=1}^{n} b_k$, and $a_1, ..., a_m \in b_i S \cap R$ implies that $P = b_i T$ and that $a_j \in b_i T$ for all j. It follows that $(a_j/b_i) \in T$ for all j, and so $S = T[(a_1/b_i)/(b_2 \cdots b_n), ..., (a_m/b_i)/(b_2 \cdots b_n)]_{\mathcal{N}'}$, where $\mathcal{N}' := \mathcal{N} \cap T[(a_1/b_i)/(b_2 \cdots b_n), ..., (a_m/b_i)/(b_2 \cdots b_n)]$. By the induction hypothesis, there exist at most $n-2$ distinct local UFDs in any chain ordered by domination strictly between T and S. Since T was arbitrary, the result follows. ▮

We conclude this section with an aside on the structure of birational local UFDs, $S > R$, where S is a spot over R obtained by adjoining a single element of q.f.(R). In general the study of local UFDs seems to be very difficult, so these results may be of general interest. Although the results here may very well be already known to the expert, we include then for lack of any reference known to the author. Both the proof of Theorem 1.9 and Example 1.11 below are due to W. Heinzer.

THEOREM 1.9. *If $(R, \mathcal{M})$ is a local UFD and $(S, \mathcal{N}) := R[a/b]_{(\mathcal{M}, a/b)}$ is another local UFD, where* $\mathrm{ht}(a, b) = 2$, *then $(a + rb)$ is a prime ideal of R for all $r \in \mathcal{M}$.*

Proof. As $S = R[(a+rb)/b]_{(\mathcal{M}, a+rb/b)}$, it suffices to show that aR is prime. We first need a claim.

Claim. $a/b \in \mathcal{N} \setminus \mathcal{N}^2$. (We have $S \cong R[T]_{(\mathcal{M}, T)}/(bT - a)$, since a and b are relatively prime in R [2, Prop. 2]. Then $a/b \in \mathcal{N}^2$ if and only if $T \in (\mathcal{N}')^2 \pmod{bT - a}$, where $\mathcal{N}' := (\mathcal{M}, T)R[T]_{(\mathcal{M}, T)}$. But $T \in (\mathcal{N}')^2 \pmod{bT - a}$ implies that $T \in T^2(R/\mathcal{M})[T]$, a contradiction.)

Now suppose by way of contradiction that a factors in R, say $a = a_1 a_2$, with both a_1 and a_2 non-units in R. Then we have two cases to consider: either there exist two distinct height 1 primes, say P_1 and P_2, of R which include a, or else there exists a height 1 prime P of R such that $a \in P^2$.

In the first case, set $V_i := R_{P_i}$, a DVR, $i = 1, 2$. If v_i is the associated valuation, then, as a and b have no common factors in R, $v_i(a/b) > 0$. Thus $R[a/b] \subseteq V_i$. Moreover, if $f \in R[a/b] \setminus (\mathcal{M}, a/b) R[a/b]$, then f can be written as a polynomial in a/b with coefficients in R whose constant term is a unit in R. Then the properties of valuations imply that $v_i(f) = 0$, and it follows that $S \subseteq V_i$. Since the maximal ideal of V_i contracts to a height 1 prime of R it must also contract to a height 1 prime of S. Call this Q_i. Putting all this together, we conclude that $a/b \in Q_1 \cap Q_2 = Q_1 Q_2$, since these are principal primes. But now we have $a/b \in \mathcal{N}^2$, in contradiction to our claim above.

Alternatively, if $a \in P^2$ then we can again define a DVR $V := R_P$ and conclude that $S \subseteq V$. Now $a/b \in \mathcal{M}_V^2$, where $\mathcal{M}_V$ denotes the maximal ideal of V. Therefore $a/b \in Q^{(2)}$, where $Q := \mathcal{M}_V \cap S$. But again, since Q is principal, we have $Q^{(2)} = Q^2$, and this also contradicts the claim. ∎

Theorem 1.9 shows that in the case that S is a birational extension of a local UFD $(R, \mathcal{M})$ which is obtained by adjoining a single element, a/b, of the quotient field and localizing at the maximal ideal generated by $\mathcal{M}$ and a/b, then S can be a UFD only when the numerator is irreducible in R. This leads to the following interesting corollary.

COROLLARY 1.10. *If* $(R, \mathcal{M})$ *is a local UFD, and if* $(S, \mathcal{N}) := R[a/b]_{(\mathcal{M}, a/b)}$ *is another local UFD, where* $\mathrm{ht}(a, b) = 2$, *then* a/b *is prime in* S.

Proof. Since $S \cong R[T]_{(\mathcal{M}, T)}/(bT - a)$, a/b is prime in S if and only if T is prime in $R[T]_{(\mathcal{M}, T)}/(bT - a)$. To check this, we mod out by T, obtaining a ring isomorphic to R/aR. The latter is an integral domain by the theorem. ∎

Considering this situation from a slightly different point of view, we might wonder whether, in the extension $S > R$ of local UFDs as above, the denominator b need be irreducible in S if it is irreducible in R. The following example demonstrates that this is not so.

EXAMPLE 1.11. Let $(R, \mathcal{M}) := k[X, Y, Z]_{(X, Y, Z)}$, where k is an arbitrary field and X, Y, and Z are indeterminates. Set $S := R[a/b]_{(\mathcal{M}, a/b)}$, where $a := X$, and $b := X + YZ$. Then $S \cong R[T]_{(\mathcal{M}, T)}/(bT - a)$ is a RLR and hence a UFD, as $bT - a$ is a regular parameter in the RLR $R[T]_{(\mathcal{M}, T)}$. However, $X/YZ = (1 - (X/(X + YZ))^{-1} - 1 \in S$, which implies that b factors in S, namely, $b = X + YZ = YZ((X/YZ) + 1)$.

2. QUADRATIC TRANSFORMS OF REGULAR LOCAL RINGS

We now explore the structure of a pair $(S, \mathcal{N}) > (R, \mathcal{M})$ of 3-dimensional RLRs where S is a quadratic transform of R, i.e., $S = R[\mathcal{M}/x]_P$, where x is a regular parameter in R and $P \in \mathrm{Spec}(R[\mathcal{M}/x])$. We begin by showing that there can exist only one RLR strictly between R and S and then we attempt to understand what sort of extension such an intermediate ring might be. We need a few lemmas concerning the irregular locus. Let us first recall the following results.

LEMMA 2.1. *Let* $(S, \mathcal{N})$ *be an* m*-dimensional local domain with* $m > 0$. *Fix* d, $0 \leqslant d \leqslant m - 1$. *Then* $\cap \{P \in \mathrm{Spec}\, S \mid \mathrm{ht}\, P = d\} = (0)$.

Proof. This is clear in $\dim S = 1$ so, without loss of generality, $m > 1$. Suppose the lemma to be false and choose $x \in \cap \{P \in \mathrm{Spec}\, S \mid \mathrm{ht}\, P = d\} \setminus \{0\}$. Use prime avoidance to choose $x_2 \in \mathcal{N} \setminus \cup \mathrm{Min}(S/(x))$. Then $\mathrm{ht}(x, x_2) = 2$, and $\mathrm{ht}(x_2) = 1$. Continue this process $d - 1$ times, getting at the ith stage a sequence such that $\mathrm{ht}(x, x_2, ..., x_{i+1}) = i + 1$, and $\mathrm{ht}(x_2, ..., x_{i+1}) = i$. Then any prime ideal P minimal over $(x, x_2, ..., x_d)$ has height d, but $x \notin P$, a contradiction. ∎

PROPOSITION 2.2 (Heinzer). *If* $(S, \mathcal{N})$ *is a local domain birationally dominating the quasilocal domain* $(D, \mathcal{P})$ *then* $\dim S \leqslant \dim D$.

Proof. We lose nothing by assuming that D is finite-dimensional, say $\dim D =: n$. We induct on this number.

$n=0$. D is its own quotient field, so there's nothing to show.

$n>0$. Let $m := \dim S$. By Lemma 2.1 we conclude that $\bigcap\{P \in \operatorname{Spec} S \mid \operatorname{ht} P = m-1\} = (0)$. Therefore there exists $P \in \operatorname{Spec} S$ such that $\mathscr{P}$ is not contained in P, $\operatorname{ht} P = m-1$. Thus $\operatorname{ht}(P \cap D) < n$. The statement now follows by localization and induction. ∎

The following lemma and its corollary tell us how to compare, in certain circumstances, the irregular locus of two different extensions. They are needed in the proof of the first principal result of this section, Theorem 2.6 below, but may have some interest in their own right.

LEMMA 2.3. *If* q.f.$(R) \supseteq S \supseteq T \supseteq R$ *are integral domains with* T *Noetherian satysfying Serre's condition* R_1 *such that* $\operatorname{Irr}(S, T)$ *is a pure codimension* 1 *closed suscheme of* Spec S, *then* $\operatorname{Irr}(S, T) \subseteq \operatorname{Irr}(S, R)$.

Proof. It suffices to show that $P \in \operatorname{Irr}(S, R)$ for every height one prime $P \in \operatorname{Irr}(S, T)$. We have that $\operatorname{ht}(P \cap T) > 1$, but $\operatorname{ht}(P \cap R) \geqslant \operatorname{ht}(P \cap T)$ (this follows after localization from Proposition 2.2), hence $P \in \operatorname{Irr}(S, R)$. ∎

The essence of the lemma is that the irreducible components of $\operatorname{Irr}(S, T)$ are a subset of those of $\operatorname{Irr}(S, R)$. Thus if the latter has a unique irreducible component then we must have equality. Since we know this to be the case if $S > R$ are RLRs with S a monoidal transform or Sally extension of R, we immediately get the following corollary.

COROLLARY 2.4. *If* q.f.$(R) \supseteq S \supseteq T \supseteq R$ *are integral domains with* T *Noetherian satisfying* R_1 *such that* $Irr(S, R)$ *is a pure codimension* 1 *irreducible closed subscheme of* Spec S *and such that* $\operatorname{Irr}(S, T)$ *is a pure codimension* 1 *closed subscheme of Spec* S, *then* $\operatorname{Irr}(S, T) = \operatorname{Irr}(S, R)$ *(as sets). In particular, this is true if* $S > R$ *are RLRs with* S *a monoidal transform or a Sally extension of* R *and if* T *is a local UFD*, $T \neq S$.

Let us also recall the following result of Sally's, as it is used frequently in the sequel. A sort of generalization of this result will be given below in Corollary 3.5.

THEOREM 2.5 [10, Corollary 2.6]. *If* $(S, \mathcal{N}) > (R, \mathcal{M})$ *are d-dimensional RLRs, and if* $V > S$, *where* V *denotes the* $\mathcal{M}$*-adic valuation ring of* R, *then* $S = R$. ∎

THEOREM 2.6. *If* $(S, \mathcal{N}) > (R, \mathcal{M})$ *are* 3*-dimensional RLRs with* S *a quadratic transform of* R. *and if* $S > T_1 > T_2 > R$, *with* $T_1 \neq S$ *and* $T_2 \neq R$, T_1 *a local UFD, and* T_2 *a RLR, then* $T_1 = T_2$.

Proof. We write $S = R[\mathscr{M}/x]_P$, where x is a regular parameter for R. Note that $\operatorname{Irr}(S, R) = V(x^2 S)$ by Proposition 1.5, since the element b occurring in the proof of that proposition is exactly x^2 in our present situation.

We claim that $S > T_2$ is actually a minimal extension of local UFDs. First we see that $\mathscr{M}_2 \nsubseteq xS$ (here $\mathscr{M}_2$ denotes the maximal ideal of T_2) because S_{xS} is the valuation ring of the $\mathscr{M}$-adic order valuation of the ring R, so $\mathscr{M}_2 \subseteq xS$ would imply that $T_2 = R$ by Theorem 2.5. In addition, $\operatorname{Irr}(S, T_2) = \operatorname{Irr}(S, R)$ by Corollary 2.4. Putting this together, we conclude that $\operatorname{ht}(xS \cap T_2) = 2$. Since $S = T_2[\mathscr{M}/x]_{\mathscr{N} \cap T_2[\mathscr{M}/x]}$, we can apply Lemma 1.7 in order to conclude that $S > T_2$ is a minimal extension of local UFDs. ▮

In addition to counting the number of local UFDs in a chain between R and S, the proof actually yields information on the nature of the rings T between. We record this as:

COROLLARY 2.7. *If $(S, \mathscr{N}) > (T, \mathscr{P}) > R$ are RLRs of the same dimension with S a quadratic transform of R, and $T \neq R$, then S is not a quadratic transform of T.*

Proof. By Corollary 2.4 we have that $\operatorname{Irr}(S, T) = \operatorname{Irr}(S, R) = V(xS)$ (as a set), where x is a regular parameter in R. If S is a quadratic transform of T, then we have that $S = T[\mathscr{P}/b]_{\mathscr{N} \cap T[\mathscr{P}/b]}$, with b a regular parameter in T. It follows that $b \in xS$. On the other hand, we known that in the case of a quadratic transform we must have $bS = \mathscr{P}S$, a height 1 prime in $\operatorname{Irr}(S, T)$. But then $bS = xS$, so $\mathscr{P} \subseteq xS$, and the valuation ring of the $\mathscr{M}$-adic order valuation of the ring R dominates T, so Sally's theorem (Theorem 2.5) now implies that $T = R$, a contradiction. ▮

Corollary 2.7 will be somewhat generalized in Section 3. For the moment we proceed to our second principal result in this section, Theorem 2.8. This theorem gives a complete answer to Question 3 of the introduction in the case that $S > R$ are RLRs having the same residue field, with S a quadratic transform of R.

THEOREM 2.8. *Let $(S, \mathscr{N}) > (T, \mathscr{P}) > (R, \mathscr{M})$ be 3-dimensional RLRs having the same residue field (e.g., this is always the case if $R/\mathscr{M}$ is algebraically closed) such that S is a quadratic extension of R, and $T \neq R, S$. Then S is necessarily a simple monoidal transform of T (i.e., S is obtained from T by blowing up a* height 2 *regular prime).*

Proof. The assumption that the rings possess the same residue field allows us to write, after a suitable change of coordinates, $S = R[y/x, z/x]_{(x, y/x, z/x)}$, where $\{x, y, z\}$ is a regular system of parameters (r.s.o.p.) for R. Now the yR-adic valuation (i.e., the normalized valuation determined by the DVR R_{yR}) is non-negative on S and has center $(y/x)\,S$, and it follows that $(y/x)\,S \cap R = yR$. Similarly, we conclude that $(z/x)\,S \cap R = zR$. Thus we must have that $(y/x)\,S \cap T$ and $(z/x)\,S \cap T$ are height 1, hence principal, prime ideals.

The generator of $(y/x)\,S \cap T$ divides y and is in turn divisible by y/x (in S), and similarly for the generator of $(z/x)\,S \cap T$. Hence these principal generators (in T) must be either $u(y/x)$ or y for the first ideal (u is a unit in S) and either $v(z/x)$ or z for the second (v is a unit in S). Applying ZMT [8; (37.4)] to $S > T$, we conclude that both $u(y/x)$ and $v(z/x)$ in T implies that $S = T$, as $\{x, u(y/x), v(z/x)\}$ is a r.s.o.p. for S in this case.

By the same token, we see that the PR_P-adic valuation, where $P := (y, z)\,R$, has center $(y/x, z/x)\,S =: Q$ on S. It follows that $S_Q = T_{Q \cap T} = R_P$, and hence $S/Q > T/(Q \cap T) > R/P$. But since the top and bottom of this chain are DVRs, we conclude that the middle is also, so $S \cap T$ must be a height 2 regular prime of T which does not include x. Thus we can find η and ζ which are minimal generators of $Q \cap T$ so that $\{x, \eta, \zeta\}$ is a r.s.o.p. for T.

Claim 1. One of η and ζ must have order 1 in the $\mathcal{M}$-adic order valuation.

(Suppose not. Then in S we may write:

$$\eta = u_1(y/x)^2 + u_2 x(y/x) + u_3 x(z/x) + u_4(y/x)(z/x) + u_5(z/x)^2 + \text{higher terms},$$

and

$$\zeta = v_1(y/x)^2 + v_2 x(y/x) + v_3 x(z/x) + v_4(y/x)(z/x) + v_5(z/x)^2 + \text{higher terms},$$

where the u_i's and v_i's are either units in S or are zero. Now $y \in Q \cap T$ implies that one of u_2 and v_2 must be non-zero. Without loss of generality we assume that $u_2 \neq 0$. The assumption that S and T have the same residue field allows us to assume (perhaps by changing the higher terms) that u_2 is in T. Then multiplying η by u_2^{-1} changes nothing essential so, without loss of generality, $u_2 = 1$. Hence, if v_2 is a unit in S (and hence, as we may assume, in T), we may replace ζ by $\zeta - v_2\eta$ in order to assume that v_2 is actually zero. Having made these assumptions, $z \in Q \cap T$ implies that v_3 must be a unit, assumed in T. Again we may then assume that $v_3 = 1$ and that $u_3 = 0$. We now have the following expressions (after changing notation):

$$\eta = x(y/x) + u_1(y/x)^2 + u_2(y/x)(z/x) + u_3(z/x)^2 + \text{higher terms},$$

and

$$\zeta = x(z/x) + v_1(y/x)^2 + v_2(y/x)(z/x) + v_3(z/x)^2 + \text{higher terms}.$$

Now if $y = t_1\eta + t_2\zeta$, $t_1, t_2 \in T$, then an examination of the expansions above shows that we must have t_1 a unit in T and $t_2 \in \mathscr{P}$, and similarly, $z = t_3\eta + t_4\zeta$, where $t_3, t_4 \in T$, t_4 a unit in T, and $t_3 \in \mathscr{P}$. Therefore x, y, and z are linearly independent mod $\mathscr{P}^2$ (over the common residue field), which implies by ZMT that $R = T$, the contradiction which we sought.)

At this point we may assume without loss of generality that η has order 1 in S and hence we may furthermore assume that $(y/x, z/x)\, S = (\eta, z/x)\, S$. Therefore $\{x, \eta, z/x\}$ is a r.s.o.p. for S. Then it follows that $(z/x) \cap T = zT$, for the alternative choice would, again by ZMT, force $T = S$.

Claim 2. $\{x, \eta, z\}$ is a r.s.o.p. in T.

(In the paragraph immediately preceding Claim 1, we saw that $\{x, \eta, \zeta\}$ is a r.s.o.p. for T. Any relation of the form: $u_1x + u_2\eta + u_3z \in \mathscr{P}^2$, where the u_i's are units or zero in T, not all zero, is itself a non-trivial relation in S. As $u_3z \in \mathscr{N}^2$, and as x and η are part of a r.s.o.p. for S, we must have that $u_1 = u_2 = 0$. Then we may assume that $u_3 = 1$, and so $z \in \mathscr{P}^2$. Hence, if the claim fails, we can write z in the form

$$(x(z/x) = \,)\; z = u_1x\eta + u_2\eta^2 + u_3x\zeta + u_4\eta\zeta + u_5\zeta^2 + \text{higher terms},$$

in T, where the u_i are either units in T or zero. We consider this as an expansion in S and take leading forms (in $\mathrm{gr}(S, \mathscr{N})$, using $\{x, \eta, z/x\}$ as a r.s.o.p. for S). Thus the $x(z/x)$-term occurring on the left-hand side must also occur on the right-hand side, and this can only happen if ζ has a (z/x)-term in its leading form. But then the presence of the $\eta\zeta$-term and the ζ^2-term on the right-hand side leads to an (uncancellable) $\eta(z/x)$-term and an (uncancellable) $(z/x)^2$-term, respectively, in the leading form of the right-hand side, neither of which occurs on the left. Thus these terms cannot occur and then neither can the η^2-term occur. Then we may replace ζ by $u_1\eta + u_3\zeta$ (in T) in order to conclude that the leading form of (z/x) and that of ζ (in $\mathrm{gr}(S, \mathscr{N})$) are one and the same. This leads to the result that $\{x, \eta, \zeta\}$ is a r.s.o.p. for S, again contradicting ZMT.)

Finally, letting $T' := T[z/x]_{(\mathscr{N} \cap T[z/x])}$, we get that $T < T' < S$, T' a RLR, so $T' = S$ by Theorem 2.6, and S is a simple monoidal transform of T. ∎

We note one corollary from the proof.

COROLLARY 2.9. *Let* $(S, \mathscr{N}) > T > (R, \mathscr{M})$ *be* 3-*dimensional RLRs having the same residue field, with* S *a quadratic transform of* R, *such that* $T \neq R, S$. *Then* $\mathscr{M}T$ *is always a* height 2 *regular prime of* T.

Proof. Using the notation of Theorem 2.8, we have that there exists $\{x, y, z\}$ a r.s.o.p. in R such that x, z form part of a r.s.o.p. in T. Hence

$(x, z)\,T$ is a regular height 2 prime ideal in T which is contained in $\mathcal{M}T$. Now ZMT implies that ht $\mathcal{M}T \leqslant 2$, so altogether we arrive at the conclusion that $\mathcal{M}T = (x, z)\,T$. ∎

One wonders whether even more structure can be found in this rather basic scenario. The following two questions, for example, are evident.

QUESTION 2.10. *Does Theorem* 2.8 *hold withut the assumption that R and S possess the same residue field*?

QUESTION 2.11. *With the hypotheses as in Theorem* 2.8, *is it always the case that T is a simple monoidal transform of R*?

The proof of Theorem 2.8 shows clearly that Question 2.11 could easily be answered affirmatively if one knew that uy (in the notation of the proof of Theorem 2.8) was actually an element of R, for then one would attain T from R simply by blowing up the ideal in R generated by x and uy. But is not at all clear to the author that this need be the case. However, we do get another corollary to the theorem in this vein.

COROLLARY 2.12. *With the notation as in the proof of Theorem* 2.8, *if we assume in addition to the assumptions in Theorem* 2.8 *that* $(y/x)\,S \cap T = u(y/x)$, *u a unit in S, then T is a simple monoidal transform of R*.

Proof. In this case the proof of Theorem 2.8 shows that x, $u(y/x)$, and z form a r.s.o.p. for T. (For x and $u(y/x)$ form part of a r.s.o.p. in S, so a non-trivial relation among x, $u(y/z)$, and z in T would lead to the conclusion that $z \in (\mathcal{M}_T)^2$ (here $\mathcal{M}_T$ denotes the maximal ideal of T); the proof of Claim 2 above shows that this is impossible.) As in the proof of Corollary 2.9, we also have that $\mathcal{M}T = (x, z)\,T$, and hence we can write $y = t_1 x + t_2 z$, $t_1, t_2 \in T$. Now taking leading forms (in $\mathrm{gr}(S, \mathcal{N})$, using $\{x, y/x, z/x\}$ as a r.s.o.p. for S), we conclude that t_2 is in $\mathcal{M}_T$ and that t_1 can be written in S as $y/x +$ higher terms in S. From this it is immediate that t_1 must have order 1 in T, so we may expand t_1 in T as $t_1 = u_1 x + u_2 u(y/x) + u_3 z +$ higher terms in T, where now the u_i's are units in T (or zero). Then comparing these two expansions of t_1, we conclude that $u_1 = 0$, and that $u_2 u(y/x) = y/x$, so that $u_2 = u^{-1}$ is in T, and hence y/x is as well. Then from the remark just before this corollary, as $y \in R$, T is necessarily a simple monoidal transform of R, namely, $T = R[y/x]_{(\mathcal{N} \cap R[y/x])}$. ∎

3. OTHER UNIFORM BOUNDS

In this final section we return to Question 1 of the introduction for RLRs, producing two alternative uniform bounds on chains of RLRs between $S > R$. Although these results are rather weaker than those given in [6, Corollary 1.11], they have the merit of working for arbitrary RLRs without the assumption of excellence. We include these results in order to provide further viewpoints on the problem.

Our first result here is on "monomial" RLR extensions (e.g., local rings on nonsingular toroidal birational varieties). The situation we are considering is that in which $\{x_1, ..., x_d\}$ is a regular system of parameters for S and there exists a regular system of parameters for R of the form $\{m_1, ..., m_d\}$, where $m_i = \prod (x_j)^{a_{ij}}$ is a monomial in the x_j's. Birationality now forces the matrix of exponents (a_{ij}) to have determinant ± 1. We say that R is a *monomial sub-RLR of S.*

LEMMA 3.1. *If $S > T > R$ are d-dimensional RLRs such that R and T are monomial sub-RLRs of S with respect to the same regular system of parameters in S, then R is a monomial sub-RLR of T.*

Proof. (a_{ij}) is invertible and it follows that $x_1, ..., x_d$ can be written as monomials in $m_1, ..., m_d$ (allowing negative exponents) by using the entries of $(a_{ij})^{-1}$ to get the exponents. Now if $\{t_1, ..., t_d\}$ are monomials in the x_j's forming a regular system of parameters for T, simple substitution shows that the t_k's can now be written as monomials in the m_i's (allowing negative exponents) and so by inverting the latter matrix, we can write the m_i's monomials in the t_k's (with necessarily positive exponents, since $R \subseteq T$). ∎

THEOREM 3.2. *If $S > R$ are d-dimensional RLRs, R a monomial sub-RLR of S, then there exists a positive integer B depending only on R and S such that whenever $S > S_1 > \cdots > S_m > R$ is a strictly descending chain of monomial sub-RLRs of S between R and S, then $m \leqslant B$. (We are assuming here the existence of a fixed regular system of parameters in S making all of the smaller rings monomial.)*

Proof. Let $(a_{ij}^{(r)})$ denote the matrix of exponents between S_{r-1} and S_r (here $S_0 := S$) and (b_{ij}) be the matrix of exponents between S and R. All of these are invertible matrices with non-negative integer entries. Consideration of the proof of Lemma 4.1 shows that we must have

$$(b_{ij}) = \prod_{k=1}^{m} (a_{ij}^{(k)}).$$

But multiplying two matrices of this sort produces a new matrix, each of whose entries is never smaller than the corresponding entry in either factor matrix. Thus if we impose the order $(c_{ij}) \leqslant (d_{ij})$ if and only if $c_{ij} \leqslant d_{ij}$ for all c_{ij}, we conclude that the fixed entries (b_{ij}) form a uniform bound on the number of multiplications possible and hence on the number of RLRs in the chain. ∎

Unfortunately, it need not be the case that if $S > T > R$ are d-dimensional RLRs and R is a monomial sub-RLR of S then T is also a monomial sub-RLR of S, as the next example illustrates.

EXAMPLE 3.3. *Let* $(S, \mathcal{N})$ be a 3-dimensional RLR with regular parameters x, y, and z, and let $(R, \mathcal{M})$ be a 3-dimensional RLR with $S > R$ and with $\{x, xy, xz\}$ a regular system of parameters for R (e.g., start with R arbitrary and let S be a quadratic transform of R with the same residue field). Letting

$$T := R\left[\frac{xy + (xz)^2}{x}\right]_{(m,\ (xy + (xz)^2)/x)}$$

we get that $S > T > R$, $\{x, y + xz^2, xz\}$ is a regular system of parameters for T, and R is a monomial sub-RLR of S; but T is a 3-dimensional RLR which is not a monomial sub-RLR of S.

Our second circle of results is based on the theorem of Sally's quoted above (Theorem 2.5) and generalizes some of the results in Section 2. We begin with yet another theorem on the structure of monoidal extensions of RLRs.

PROPOSITION 3.4. *Let $(S, \mathcal{N}) > (R, \mathcal{M})$ be d-dimensional RLRs, where S is a monoidal transform of R with center Q, and let T be any RLR with $S > T > R$. Let P denote the center of the QR_Q-adic valuation on T. Then the valuation ring of the PT_P-adic valuation is the same as that of the QR_Q-adic valuation.*

Proof. By hypothesis, we can write S as a localization of $R[Q/x]$, where $x \in Q$ is a regular parameter in R which is not a unit in S. Thus the valuation ring of the QR_Q-adic valuation is exactly S_{xS}. On the other hand, the valuation ring of the PY_P-adic valuation may be written as $T_P[PT_P/x]_{xT_P[PT_P/x]} = T_P[P/x]_{xT_P[P/x]} = T[P/x]_{xT[P/x]}$. $(P/x) \subseteq S$ by hypothesis, so $T[P/x] \subseteq S$, but $Q \subseteq P$ implies further that $R[Q/x] \subseteq T[P/x] \subseteq S$. Thus $S = T[P/x]_{(\mathcal{N} \cap T[P/x])}$ since $S = R[Q/x]_{(\mathcal{N} \cap R[Q/x])}$. Now if V is the valuation ring of the PT_P-adic valuation, then V has center $xT[P/x]$ on $T[P/x]$ and hence has center xS on S, which implies that $S_{xS} \subseteq V$, so $S_{xS} = V$. ∎

The following corollary is essentially the heart of the matter here.

COROLLARY 3.5. *Let $S > T_1 > T_2 > R$ be d-dimensional RLRs such that S is a monoidal transform of R, with center Q. Let P_1 denote the center of* *the QR_Q-adic valuation on T_1, and let P_2 denote the center of the QR_Q-adic valuation on T_2. Then* ht $P_1 <$ ht P_2.

Proof. We necessarily have that ht $P_1 \leqslant$ ht P_2 (e.g., by Proposition 2.2). By the theorem, the QR_Q-adic valuation ring and the $P_2(T_2)_{P_2}$-adic valuation ring are the same. Denote this by V. We have that $V > (T_1)_{P_1} > (T_2)_{P_2}$. If ht $P_1 =$ ht P_2, then Sally's theorem (Theorem 2.5) now implies that $(T_1)_{P_1} = (T_2)_{P_2}$, so $P_1 \notin \operatorname{Irr}(T_1, T_2)$.

On the other hand, every height 1 prime ideal in $\operatorname{Irr}(T_1, T_2)$ must be contained in $QR_Q \cap T_1 = P_1$, by Lemma 1.3. Thus the closedness of the locus of primes irregular for the extension $T_1 > T_2$ implies that $P_1 \in \operatorname{Irr}(T_1, T_2)$, a contradiction. ∎

By this corollary, the height of the center of the QR_Q-adic valuation must drop for each RLR as we move up a chain between S and R. This immediately leads to our new uniform bound in the case that S is a monoidal transform of R.

COROLLARY 3.6. *If $S > T_1 > T_2 > \cdots > T_n > R$ are distinct d-dimensional RLRs such that S is a monoidal transform of R with center Q, then* $n \leqslant (\text{ht } Q) - 2$.

The uniform bound in Corollary 3.6 has already been found in [6, Proposition 2.3] under the further assumption that R is excellent. The advantage of the present approach is not only that it disposes of this assumption, but also that this new proof seems to yield a more intrinsic explanation as to why this uniform bound occurs.

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